

Spectrum of stochastic evolution operators: polynomial basis approach

C.P.Dettmann and Niels S ndergaard

Northwestern University, Department of Physics & Astronomy
2145 Sheridan Road, Evanston, Illinois 60208 USA

Gergely Palla and G bor Vattay

Department of Physics of Complex Systems, E tv s University
P zm ny P ter s t ny 1/A, H-1117 Budapest, Hungary

(February 1, 2008)

The spectrum of the evolution operator associated with a nonlinear stochastic flow with additive noise is evaluated by diagonalization in a polynomial basis. The method works for arbitrary noise strength. In the weak noise limit we formulate a new perturbative expansion for the spectrum of the stochastic evolution operator in terms of expansions around the classical periodic orbits. The diagonalization of such operators is easier to implement than the standard Feynman diagram perturbation theory. The result is a stochastic analog of the Gutzwiller semiclassical spectral determinant with the “ \hbar ” corrections computed to at least two orders more than what has so far been attainable in stochastic and quantum-mechanical applications, supplemented by the estimate for the late terms in the asymptotic saddlepoint expansions.

02.50.Ey, 03.20.+i, 03.65.Sq, 05.40.+j, 05.45.+b

I. INTRODUCTION

The periodic orbit theory relates the spectrum of the Fokker-Plank operator and its weighted evolution operator generalizations to the periodic orbits via trace formulas, dynamical zeta functions and spectral determinants [3,4]. For quantum mechanics the periodic orbit theory is exact on the semiclassical level [13], whereas the quintessentially quantum effects such as creeping, tunneling and diffraction have to be included as corrections. In particular, the higher order \hbar corrections can be computed perturbatively by means of Feynman diagrammatic expansions [6].

The evolution operator formalism allows us to calculate long time averages in a chaotic system in terms of the eigenvalues of evolution operators. The simplest example is provided by the Perron-Frobenius operator

$$\mathcal{L}\rho(x') = \int dx \delta(f(x) - x')\rho(x)$$

for a *deterministic* map $f(x)$ which maps a density distribution $\rho(x)$ forward in time. Our purpose here is to develop effective methods for computation of spectra of *stochastic* evolution operators. In case at hand, already a discrete time 1-dimensional discrete Langevin equation [7,8],

$$x_{n+1} = f(x_n) + \sigma\xi_n, \quad (1)$$

with ξ_n independent normalized random variables, suffices to reveal the structure of the dependence on the noise.

We treat a chaotic system with external noise by replacing the deterministic evolution δ -function kernel by the Fokker-Planck kernel corresponding to (1), a sharply peaked noise distribution function

$$\mathcal{L}(x', x) = \delta_\sigma(f(x) - x'). \quad (2)$$

In ref. [1] we have treated the problem of computing the spectrum of this operator by standard field-theoretic Feynman diagram expansions; in ref. [2] we offered a more elegant formulation of the perturbation theory in terms of smooth conjugacies. This time we evaluate the evolution operator in an explicit polynomial basis. The procedure, which is relatively easy to automatize, enables us to go three orders further in the perturbation theory; in the language of Feynman diagrams of ref. [1], in the new approach developed below we are able to compute the perturbative corrections up 5-loop level, as well as study the asymptotics of late terms in the perturbative expansion.

The paper is organized as follows: in refsectFLOWS we review the evolution operator formalism for smooth flows. In sect. II we explain the theorems that guarantee that spectral determinants for Axiom A systems are entire. In refsectNUMERICAL we discuss the numerical tests of our perturbative expansions.

II. MATRIX REPRESENTATION OF Perron-Frobenius operator

We shall sketch here the basic ideas behind the proofs that the classical spectral determinants are entire, without burdening the reader with too many technical details (rigorous treatment is given in refs. [19,20,10]). The main point is that the spectral determinants are entire functions in any dimension, provided that

1. the evolution operator is *multiplicative* along the flow,
2. the symbolic dynamics is a *finite subshift*,
3. all cycle eigenvalues are *hyperbolic* (sufficiently bounded away from 1),
4. the map (or the flow) is *real analytic*, *ie.* it has a piecewise analytic continuation to a complex extension of the phase space.

As in physical applications one studies smooth dynamical observables, we restrict the space that \mathcal{L} acts on to smooth functions. In practice “real analytic” means that all expansions are polynomial expansions. In order to illustrate how this works in practice, we first work out a simple example.

A. Expanding maps with a single fixed point

We start with the trivial example of a repeller with only one expanding linear branch [4]

$$f(x) = \Lambda x \quad |\Lambda| > 1 .$$

The action of the associated deterministic, noiseless Perron-Frobenius operator is

$$\mathcal{L}\phi(y) = \int dx \delta(y - \Lambda x) \phi(x) = \frac{1}{|\Lambda|} \phi(y/\Lambda) .$$

From this one immediately identifies the eigenfunctions and eigenvalues:

$$\mathcal{L} y^n = \frac{1}{|\Lambda| \Lambda^n} y^n, \quad n = 0, 1, 2, \dots \quad (3)$$

We note that the eigenvalues Λ^{-n-1} fall off exponentially with n , and that the trace of \mathcal{L} is given by

$$\text{tr } \mathcal{L} = \frac{1}{|\Lambda|} \sum_{n=0}^{\infty} \Lambda^{-n} = \frac{1}{|\Lambda|(1 - \Lambda^{-1})} = \frac{1}{|f' - 1|} . \quad (4)$$

A similar result is easily obtained for powers of \mathcal{L} , and for the spectral determinant one obtains:

$$\det(1 - z\mathcal{L}) = \prod_{k=0}^{\infty} \left(1 - \frac{z}{|\Lambda| \Lambda^k} \right) = \sum_{k=0}^{\infty} Q_k t^k ,$$

$t = -z/|\Lambda|$, where the cumulants Q_k are given explicitly by the Euler formula [24]

$$Q_k = \frac{1}{1 - \Lambda^{-1}} \frac{\Lambda^{-1}}{1 - \Lambda^{-2}} \cdots \frac{\Lambda^{-k+1}}{1 - \Lambda^{-k}} . \quad (5)$$

These coefficients decay asymptotically *faster* than exponentially, as $\Lambda^{-k(k-1)/2}$. An intuitive way of comprehending the spectrum is to view the composition with an expanding map as a “smoothing” operator; fast variations in the initial density distribution $\phi(x)$, corresponding to high powers of x , are wiped out quickly by the smoothing operator.

In a suitable polynomial basis $\phi_n(x)$ the operator has an explicit matrix representation

$$(\mathcal{L}\phi)_n(x) = \sum_{m=0}^{\infty} L_{nm} \phi_m(x) .$$

In the single fixed-point example (3), $\phi_n = y^n$, and \mathcal{L} is diagonal, $\mathcal{L}_{nm} = \delta_{mn} \Lambda^{-n}/|\Lambda|$. In general case, a matrix representation can be constructed by means of Cauchy complex contour integrals.

The simplest example of how the Cauchy formula is employed is provided by a nonlinear inverse map $\psi = f^{-1}$, $s = \text{sign}(\psi')$

$$\mathcal{L}\phi(w) = \int dx \delta(w - f(x))\phi(x) = s \psi'(w) \phi(\psi(w)).$$

Assume that ψ is a contraction of the unit disk, i.e.

$$|\psi(w)| < \theta < 1 \quad \text{and} \quad |\psi'(w)| < C < \infty \quad \text{for} \quad |w| < 1,$$

and expand ϕ in a polynomial basis by means of the Cauchy formula

$$\phi(x) = \sum_{n \geq 0} x^n \phi_n = \oint \frac{dw}{2\pi i} \frac{\phi(w)}{w - x}, \quad \phi_n = \oint \frac{dw}{2\pi i} \frac{\phi(w)}{w^{n+1}}. \quad (6)$$

In this basis, \mathcal{L} is represented by the matrix

$$\mathcal{L}\phi(w) = \sum_{m,n} w^m L_{mn} \phi_n, \quad L_{mn} = \oint \frac{dw}{2\pi i} \frac{s \psi'(w) (\psi(w))^n}{w^{m+1}}. \quad (7)$$

Taking the trace and summing we get:

$$\text{tr } \mathcal{L} = \sum_{n \geq 0} L_{nn} = \oint \frac{dw}{2\pi i} \frac{s \psi'(w)}{w - \psi(w)}$$

This integral has but one simple pole at the unique fix point $w^* = \psi(w^*) = f(w^*)$. Hence

$$\text{tr } \mathcal{L} = \frac{s \psi'(w^*)}{1 - \psi'(w^*)} = \frac{1}{|f'(w^*) - 1|}$$

in agreement with (4).

In practice we do not evaluate the matrix elements (7) by Cauchy integrals; instead, we use the polynomial basis [10,11], with the left basis given by derivatives $\frac{z^m}{(m!)}$, and the right basis by polynomials $\frac{\partial^m}{\partial x^m}$. An analytic function can be written as $F(y) = \sum F_m \frac{y^m}{m!}$, and the coefficients F_m can be obtained by $F_m = \frac{\partial^m F(y)}{\partial y^m} \Big|_{y=0}$, or in other words, the dual basis of $\frac{y^m}{m!}$ is $\frac{\partial}{\partial y^m}$. So the L_{lk} matrix element is obtained by acting the operator on $\frac{y^k}{k!}$ and differentiating the result l times at $y = 0$:

$$\begin{aligned} L_{mm'}^i &= \left\langle \frac{\partial^m}{\partial y'^m} \left| \mathcal{L}_{s_i}(y', y) \right| \frac{y^{m'}}{m'!} \right\rangle \\ &= \left\langle \frac{\partial^m}{\partial y'^m} \left| \delta(y' + x_{i+1} - f(y + x_i)) \right| \frac{y^{m'}}{m'!} \right\rangle \end{aligned} \quad (8)$$

While it is not at all obvious that what is true for a single fixed point should also apply to a Cantor set of periodic points, the same asymptotic decay of expansion coefficients is obtained when several expanding branches are involved. For a two-branch repeller, the procedure is the same, except the integral (6) picks up a contribution from each branch.

From bounds on the elements \mathcal{L}_{mn} one verifies [19,20,10] that they again fall off as $\Lambda^{-k^2/2}$, concluding that the \mathcal{L} eigenvalues fall off exponentially for a general Axiom A one-dimensional map.

III. Evolution operator COMPOSITION FOR PIECEWISE-ANALYTIC EXPANDING MAPS

Suppose we can decompose the evolution operator in a sum of two operators $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$. Here we shall study a noisy map with a binary Markov partition $\mathcal{M} = \{\mathcal{M}_0, \mathcal{M}_1\}$. We distinguish two branches of the map : $f(x) = f_0(x)$ if $x \in \mathcal{M}_0$, and $f(x) = f_1(x)$ if $x \in \mathcal{M}_1$. In the case of Gaussian noise the corresponding operators are

$$\begin{aligned} \mathcal{L}_0(x', x) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x' - f_0(x))^2}, \quad x \in \mathcal{M}_0 \\ \mathcal{L}_1(x', x) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x' - f_1(x))^2}, \quad x \in \mathcal{M}_1. \end{aligned} \quad (9)$$

If you visualize \mathcal{L} as an $[2n \times 2n]$ matrix, in the weak noise limit the matrix elements for whom x is in the other partition are negligible and can be set to zero, so \mathcal{L}_0 and \mathcal{L}_1 are $[n \times 2n]$ matrices.

For piecewise-analytic maps the decomposition is exact; for stochastic operators it assumes that the overlap of the two kernels is insignificant and can be neglected. Then we can write:

$$\begin{aligned} \ln \det(1 - z(\mathcal{L}_0 + \mathcal{L}_1)) &= \text{tr} \ln(1 - z(\mathcal{L}_0 + \mathcal{L}_1)) \\ &= - \sum_{n=1}^{\infty} \frac{z^n}{n} \text{tr} (\mathcal{L}_0 + \mathcal{L}_1)^n. \end{aligned} \quad (10)$$

A contribution to this sum corresponds either to a prime cycle $p = s_1 s_2 \cdots s_{n_p}$, $s_i \in \{0, 1\}$, or to its repeat

$$\text{tr} \mathcal{L}_{p^r} = \text{tr} (\mathcal{L}_{s_1} \mathcal{L}_{s_2} \cdots \mathcal{L}_{s_{n_p}})^r.$$

Expanding the n th power we get contributions from all symbol sequences of length n :

$$\text{tr} (\mathcal{L}_0 + \mathcal{L}_1)^n = \sum_p n_p \sum_{r=1}^{\infty} \delta_{n, n_p r} \text{tr} (\mathcal{L}_p)^r, \quad (11)$$

where p denotes a prime cycle itinerary composed of 0's and 1's, n_p is the prime cycle length and r is the repetition number. For instance, $p = 011$ then $\mathcal{L}_{011} = \mathcal{L}_0 \mathcal{L}_1 \mathcal{L}_1 = \mathcal{L}_0 \mathcal{L}_1^2$. The cyclic property of trace yields the n_p factor. Substituting into (10) we obtain

$$\ln \det(1 - z(\mathcal{L}_0 + \mathcal{L}_1)) = - \sum_p \sum_{r=1}^{\infty} \frac{z^{n_p r}}{r} \text{tr} \mathcal{L}_p^r. \quad (12)$$

The sum over repeats yields a factorized formula of the dynamical zeta function type

$$\det(1 - z(\mathcal{L}_0 + \mathcal{L}_1)) = \prod_p \det(1 - z^{n_p} \mathcal{L}_p). \quad (13)$$

The operators are defined on piecewise monotonic maps, so there is only one periodic orbit for a given prime cycle itinerary.

IV. SADDLE POINT EXPANSIONS IN TERMS OF PRIME CYCLES

In the weak noise limit the kernel is sharply peaked, so it makes sense to expand it in terms of the Dirac delta function and its derivatives:

$$\begin{aligned} \delta_{\sigma}(y) &= \sum_{m=0}^{\infty} \frac{a_m \sigma^m}{m!} \delta^{(m)}(y) \\ &= \delta(y) + a_2 \frac{\sigma^2}{2} \delta^{(2)}(y) + a_3 \frac{\sigma^3}{6} \delta^{(3)}(y) + \dots \end{aligned} \quad (14)$$

where

$$\delta^{(k)}(y) = \frac{\partial^k}{\partial y^k} \delta(y),$$

and the coefficients a_m depend on the choice of the kernel. We have omitted the $\delta^{(1)}(y)$ term in the above because in our applications we shall impose the saddle-point condition, that is, we shift f by a constant to ensure that the noise peak corresponds to $y = 0$, so $\delta'_{\sigma}(0) = 0$. For example, if $\delta_{\sigma}(y)$ is a Gaussian kernel, it can be expanded as

$$\begin{aligned} \delta_{\sigma}(y) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2/2\sigma^2} = \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{n! 2^n} \delta^{(2n)}(y) \\ &= \delta(y) + \frac{\sigma^2}{2} \delta^{(2)}(y) + \frac{\sigma^4}{8} \delta^{(4)}(y) + \dots \end{aligned} \quad (15)$$

V. Evolution operatorS IN A MATRIX REPRESENTATION

In this section our goal is to calculate noise corrections to the leading eigenvalue of the Perron-Frobenius operator. Expression (10) shows, that in order to do that first we have to calculate $\text{tr } \mathcal{L}^n$, and from equation (11) we see that $\text{tr } \mathcal{L}^n$ should be generated from traces of \mathcal{L} on *periodic orbits*. This section is a breaif review of how this was carried out in practise using the matrix representation of the Perron-Frobenius operator.

If the coordinates of a prime cycle are x_1, \dots, x_{n_p} , the operator of a periodic orbit segment is

$$\mathcal{L}_{s_i}(y', y) = \sum_{m=0}^{\infty} \frac{a_m \sigma^m}{m!} \delta^m(y' + x_{i+1} - f(x_i + y)), \quad (16)$$

where $x_{n_p+1} = x_1$ and a_m are the moments of the noise. The full contribution to the trace from this periodic orbit is:

$$\text{tr } \mathcal{L}_p = \int dy_1 \dots dy_p \mathcal{L}_{s_{n_p}}(y_1, y_{n_p}) \cdots \mathcal{L}_{s_2}(y_3, y_2) \mathcal{L}_{s_1}(y_2, y_1).$$

The calculation of this contribution can be computerized if we represent (16) in a matrix form using the polynomial basis (8). The matrix elements of \mathcal{L}_{s_i} are:

$$L_{mm'}^i = \left\langle \frac{\partial^m}{\partial y'^m} \middle| \mathcal{L}_{s_i}(y', y) \middle| \frac{y^{m'}}{m'!} \right\rangle = \sum_{n=\max(m'-m, 0)}^{\infty} \frac{a_n \sigma^n}{n!} B_{m+n, m'}^{(i)}, \quad (17)$$

where the B matrix is the representation of the noiseless operator:

$$B_{kk'}^i = \left\langle \frac{\partial^k}{\partial y'^k} \middle| \delta(y' + x_{i+1} - f(y + x_i)) \middle| \frac{y^{k'}}{k'!} \right\rangle \quad (18)$$

If the Dirac-delta in B acts on $\frac{y^{k'}}{k'!}$ we get:

$$\begin{aligned} \int dy \delta(y' + x_{i+1} - f(x_i + y)) \frac{y^{k'}}{k'!} &= \frac{(f^{-1}(x_{i+1} + y') - x_i)^{k'}}{k'! |f'(f^{-1}(x_{i+1} + y'))|} \\ &= \frac{\text{sign}(f')}{(k' + 1)!} \frac{d}{dy'} (f^{-1}(x_{i+1} + y') - x_i)^{k'+1}, \end{aligned} \quad (19)$$

where $\text{sign}(f')$ is the sign of $f'(f^{-1}(x_{i+1} + y'))$. There is no summation for the branches of the preimages of f , since each branche takes us back to a differernt surrounding, so at each point the branch is choosen by the previous point of the orbit. The k, k' matrix element of B is the k th derivative of this at $y' = 0$:

$$\begin{aligned} B_{k, k'}^i &= \frac{\text{sign}(f')}{(k' + 1)!} \frac{\partial^{k+1}}{\partial y'^{k+1}} F^i(y')^{k'+1} \bigg|_{y'=0}, \\ F^i(y') &= f^{-1}(x_{i+1} + y') - x_i \end{aligned} \quad (20)$$

Let us write $F^i(y')$ in power series of y' . If $k' > k$ then after the derivates are taken every term in $B_{k, k'}$ has at least a $(y')^{k'-k}$ coefficient. We have to evaluate the derivative at $y' = 0$, so we see that B is a triangular matrix, $B_{k, k'} = 0$ if $k' > k$.

A. Multinomials

The non-zero matrix elements are [23]

$$\left(\sum_{l=1}^{\infty} \frac{x_l}{l!} t^l \right)^m = m! \sum_{n=l}^{\infty} \frac{t^n}{n!} \sum (n | a_1 a_2 \dots a_n)' x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}, \quad (21)$$

where the sum (\sum) goes over all non-negative integers such that:

$$a_1 + 2a_2 + \dots + na_n = n, \quad a_1 + a_2 + \dots + a_n = m \quad (22)$$

and the multinomial coefficient is:

$$(n|a_1a_2\dots a_n)' = \frac{n!}{(1!)^{a_1}a_1!(2!)^{a_2}a_2!\dots(n!)^{a_n}a_n!} \quad (23)$$

If we expand $F^i(y')$ in a Taylor series, the constant term is zero because of $f^{-1}(x_{i+1}) = x_i$. So we can write:

$$F^i(y') = \frac{y'}{\Lambda} + \sum_{l=2}^{\infty} \frac{F_l^i}{l!} y'^l \quad (24)$$

We apply the formula (21) to $F^i(y')$ with power $k' + 1$:

$$(F^i(y'))^{k'+1} = (k' + 1)! \sum_{n=k'+1}^{\infty} \frac{y'^n}{n!} \sum (n|a_1a_2\dots a_n)' \frac{1}{\Lambda^{a_1}} (F_2^i)^{a_2} \dots (F_n^i)^{a_n}. \quad (25)$$

In the $(k + 1)$ th derivative of this at $y' = 0$ only the $n = k + 1$ term is non-zero. So considering only one branch, with positive sign of f'_b :

$$B_{k,k'}^i = \sum (k + 1|a_1a_2\dots a_{k+1})' \frac{1}{\Lambda^{a_1}} (F_2^i)^{a_2} \dots (F_{k+1}^i)^{a_{k+1}} \quad (26)$$

$$a_1 + 2a_2 + \dots + (k + 1)a_{k+1} = k + 1 \quad (27)$$

$$a_1 + a_2 + \dots + a_{k+1} = k' + 1 \quad (28)$$

$$(k + 1|a_1a_2\dots a_{k+1})' = \frac{(k + 1)!}{(1!)^{a_1}a_1!(2!)^{a_2}a_2!\dots((k + 1)!)^{a_{k+1}}a_{k+1}!} \quad (29)$$

(Here F_l^i is the l th derivative of F^i on the particular branch which we have to choose.) For the diagonal and the nearest off-diagonals the matrix elements read as:

$$B_{mm} = \frac{1}{\Lambda^{m+1}}, \quad m = 0, 1, \dots \quad (30)$$

$$B_{m+1,m} = \frac{(m + 2)(m + 1)}{2} \frac{F_2^i}{\Lambda^m} \quad (31)$$

$$B_{m+2,m} = \frac{(m + 3)!}{8(m - 1)!} \frac{(F_2^i)^2}{\Lambda^{m-1}} + \frac{(m + 3)!}{6(m + 1)!} \frac{F_3^i}{\Lambda^m} \quad (32)$$

$$B_{m+3,m} = \frac{(m + 4)!}{48(m - 2)!} \frac{(F_2^i)^3}{\Lambda^{m-2}} + \frac{(m + 4)!}{12(m - 1)!} \frac{F_2^i F_3^i}{\Lambda^{m-1}} + \frac{(m + 4)!}{24m!} \frac{F_4^i}{\Lambda^m} \quad (33)$$

etc...,

B. Noiseless case

In the noiseless case the B^i matrices are the representations of the Perron-Frobenius operator. Since they are triangular, their eigenvalues are:

$$\lambda_m = B_{mm} = \frac{1}{\Lambda^{m+1}} \quad . \quad (34)$$

Multiplied, the triangular matrices yield a triangular matrix, and the diagonal elements of the result can be obtained simply by multiplying the two corresponding diagonal elements. This results that the trace of the \mathcal{L} on a periodic orbit is the following:

$$\begin{aligned}
\text{tr } \mathcal{L}_p &= \text{tr } L_{s_1} L_{s_2} \cdots L_{s_{n_p}} = \sum_{m=0}^{\infty} \frac{1}{|\Lambda_p|^{(m+1)}} \\
&= \frac{1}{|\Lambda_p|} \sum_{m=0}^{\infty} \frac{1}{\Lambda_p^m} = \frac{1}{|\Lambda_p| (1 - 1/\Lambda_p)} = \frac{1}{|1 - \Lambda_p|}
\end{aligned} \tag{35}$$

The absolute value is a consequence of the $\text{sign}(f')$ factor in (20). For the total trace, using (11) we get back the standard deterministic trace formula [4]

$$\text{tr } \mathcal{L}^n = \sum_p n_p \sum_{r=1}^{\infty} \delta_{n, n_p r} \frac{1}{|1 - \Lambda_p^r|} \tag{36}$$

C. Numerical tests

Here we continue the calculations of Sect. ? of ref. [2], where more details and discussion may be found. We test our perturbative expansion on the repeller of the 1-dimensional map

$$f(x) = 20 \left(\frac{1}{16} - \left(\frac{1}{2} - x \right)^4 \right). \tag{37}$$

This repeller is a nice example of an “Axiom A” expanding system of bounded nonlinearity and complete binary symbolic dynamics, for which the deterministic evolution operator eigenvalues converge super-exponentially with the cycle length.

The inverse of f has two branches:

$$f^{-1}(y) = \left[\frac{1}{2} - \left(\frac{1}{16} - \frac{y}{20} \right)^{\frac{1}{4}}, \frac{1}{2} + \left(\frac{1}{16} - \frac{y}{20} \right)^{\frac{1}{4}} \right]. \tag{38}$$

Due to the symmetry of (37) around $x = 0.5$, the derivatives on the two inverse branches have the same absolute value, but opposite signs. We compute the leading eigenvalue of the evolution operator (the repeller escape rate) in the presence of Gaussian noise, using three complementary approaches. The perturbative result in terms of periodic orbits and the weak noise corrections is compared to the eigenvalue computed by a numerical lattice discretization in ref. [1]. In the preceding paper (ref. [2]) we compared the numerical eigenvalue with the σ^4 result and estimated the coefficient of σ^6 to be approximately 2700. Here we compute the order σ^6 coefficient 2076.47... to 14 digits accuracy, as well as the σ^8 and σ^{10} coefficients. Furthermore, we estimate the asymptotic form of the σ^{2m} coefficient.

The numerical calculations of $\text{tr } \mathcal{L}^n$ proceeds as follows:

1. Creating periodic orbits up to length 10 using the method of iterating backwards [4]
2. Computing $[24 \times 24]$ B matrices at each point of orbits
3. From the B matrices computing $[16 \times 16]$ L matrices at each point of orbits, up to σ^{10} corrections in each matrix-element
4. Multiplying the L matrices. At shorter orbits repeating this r times. ($rn_p = n$)
5. Computing the traces of the result matrices, multiplying them by n_p and adding them up to get $\text{tr } \mathcal{L}^n$ as in (11)
6. Create cummulants from the traces.
7. Find zeros.

At $n = 2$ $[20 \times 20]$ sized L (and corresponding to that $[28 \times 28]$ sized B) matrices were used because of slower convergence in $\text{tr } \mathcal{L}^n$ while enlarging the matrix size. At $n = 1$ $[26 \times 26]$ is the size in L one has to reach to get decent result for $\text{tr } \mathcal{L}$.

D. From traces to σ^{2m} corrections

To get the leading eigenvalue of the Perron-Frobenius operator from the traces of \mathcal{L}^n , we follow the method outlined in [2]. The trace of \mathcal{L}^n for $n = 1, 2, \dots, 10$ can be written as:

$$\text{tr } \mathcal{L}^n = \sum_{j=0}^{\infty} C_{nj} \sigma^j, \quad (39)$$

where we know the value of the first five non-zero $C_{n0}, \dots, C_{n,10}$. The cumulants Q_{nj} in

$$\det(1 - z\mathcal{L}) = 1 - \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} Q_{nj} z^n \sigma^j \quad (40)$$

are obtained recursively as

$$Q_{nm} = \frac{1}{n} \left(C_{nm} - \sum_{k=1}^{n-1} \sum_{l=0}^m Q_{k,m-l} C_{n-k,l} \right). \quad (41)$$

For Gaussian noise only even l terms contribute. Let z_0 be the solution of the noiseless condition $\det(1 - z\mathcal{L})|_{\sigma=0} = 0$. Expanding the spectral determinant around z_0 and $\sigma^2 = 0$, we write:

$$\det(1 - (z + z_0)\mathcal{L}) = F - F_{10}z - F_{02}\sigma^2 - F_{20}z^2 - F_{12}z\sigma^2 - F_{04}\sigma^4 - \mathcal{O}(z^3, \sigma^6) \quad (42)$$

where the coefficients can be obtained from the cumulants as:

$$\begin{aligned} F &= 1 - \sum_{m=1}^n Q_{m,0} z_0^m & F_{10} &= \sum_{m=1}^n m Q_{m,0} z_0^{m-1} \\ F_{02} &= \sum_{m=1}^n Q_{m,2} z_0^m & F_{20} &= \frac{1}{2} \sum_{m=2}^n m(m-1) Q_{m,0} z_0^{m-2} \\ F_{12} &= \sum_{m=1}^n m Q_{m,2} z_0^{m-1} & F_{04} &= \sum_{m=1}^n Q_{m,4} z_0^m \end{aligned} \quad (43)$$

If we want to calculate only the σ corrections just up to fourth order, first we have to solve this equation:

$$\begin{aligned} &F - F_{10}(z_2\sigma^2 + z_4\sigma^4) - F_{02}\sigma^2 - F_{20}(z_2\sigma^2 + z_4\sigma^4)^2 \\ &- F_{12}(z_2\sigma^2 + z_4\sigma^4)\sigma^2 - F_{04}\sigma^4 = 0. \end{aligned} \quad (44)$$

Since σ is a parameter, the coefficients of σ^0 , or σ^2 or σ^4 have to vanish:

$$\begin{aligned} F &= 0 \\ -F_{10}z_2 - F_{02} &= 0 \\ -F_{10}z_4 - F_{20}z_2^2 - F_{12}z_2 - F_{04} &= 0 \end{aligned}$$

The first equation gives solution for z_0 , which numerically was done by Newton's method. From the second we can express z_2 , from the third z_4 :

$$z_2 = -\frac{F_{02}}{F_{10}} \quad z_4 = -\frac{F_{20}z_2^2 + F_{12}z_2 + F_{04}}{F_{10}} \quad (45)$$

To get the eigenvalue we have to reciprocate z :

$$\begin{aligned} \nu_0 + \nu_2\sigma^2 + \nu_4\sigma^4 &= \frac{1}{z_0 + z_2\sigma^2 + z_4\sigma^4} = \frac{1}{z_0} - \frac{z_2}{z_0^2}\sigma^2 - \left(\frac{z_4}{z_0^2} - \frac{z_2^2}{z_0^3} \right) \sigma^4 \\ &\quad + \mathcal{O}(\sigma^6) \end{aligned} \quad (46)$$

We see that:

$$\nu_0 = \frac{1}{z_0}, \quad \nu_2 = -\frac{z_2}{z_0^2}, \quad \nu_4 = -\frac{z_4}{z_0^2} + \frac{z_2^2}{z_0^3}, \quad (47)$$

or by substituting into these (45):

$$\begin{aligned}\nu_0 &= \frac{1}{z_0}, & \nu_2 &= \frac{F_{02}}{F_{10}}\nu_0^2, \\ \nu_4 &= \frac{1}{F_{10}^3}(F_{20}F_{02}^2 - F_{12}F_{02}F_{10} + F_{04}F_{10}^2 + F_{02}^2F_{10}\nu_0)\nu_0^2.\end{aligned}\tag{48}$$

To go up further in the powers of σ corrections, we have to do the same as we have done for order four, the details of this is in the appendix.

| n | ν_0 | ν_2 | ν_4 |
|-----|-------------------|------------------|-----------------|
| 1 | 0.308 | 0.42 | 2.2 |
| 2 | 0.37140 | 1.422 | 32.97 |
| 3 | 0.3711096 | 1.43555 | 36.326 |
| 4 | 0.371110995255 | 1.435811262 | 36.3583777 |
| 5 | 0.371110995234863 | 1.43581124819737 | 36.35837123374 |
| 6 | 0.371110995234863 | 1.43581124819749 | 36.358371233836 |
| n | ν_6 | ν_8 | |
| 1 | 17.4 | 168.0 | |
| 2 | 1573.3 | 112699.9 | |
| 3 | 2072.9 | 189029.0 | |
| 4 | 2076.479 | 189298.8 | |
| 5 | 2076.4770492 | 189298.12802 | |
| 6 | 2076.47704933320 | 189298.128042526 | |
| 7 | 2076.47704933321 | 189298.128042526 | |

TABLE I. Significant digits of the leading deterministic eigenvalue and its $\sigma^2, \dots, \sigma^8$ coefficients, calculated from the spectral determinant as a function of the cycle truncation length n . Note the super-exponential convergence of all coefficients. We have computed all cycles to length 10, but contributions of those longer than $n = 6$ lie below the machine precision.

The perturbative corrections to the leading eigenvalue (escape rate) of the weak-noise evolution operator are given in table I, showing super-exponential convergence with the truncation cycle length n . The super-exponential convergence has been proven for the deterministic, ν_0 part of the eigenvalue [9,10], but has not been studied for noisy kernels. It is seen that a good first approximation is obtained already at $n = 2$, using only 3 prime cycles, and $n = 6$ (23 prime cycles in all) is in this example sufficient to exhaust the limits of double precision arithmetic. The exact value of $\nu_6 = 2076.47 \dots$ is not wildly different to our previous numerical estimate [2] of 2700.

VI. TESTING THE RESULTS

In previous papers the eigenvalue was computed numerically. In fig. 2 to fig. 3 we check how well our results fit that. The present result fits the data of the numerical discretisation method well in the small noise region, but not above $\sigma = 0.1$. At given σ , a matrix representation of the Perron-Frobenius operator can be obtained numerically, at each matrix element carrying out a numerical integration:

$$L_{l,k} = \frac{\partial^l}{\partial y^l} \left[\frac{1}{\sqrt{2\pi}\sigma k!} \int dz e^{-\frac{(y-f(z))^2}{2\sigma^2}} z^k \right] \Big|_{y=0}. \quad (49)$$

The leading eigenvalue of this matrix fits well the discretisation data well at large σ , as it is shown on fig. 4.

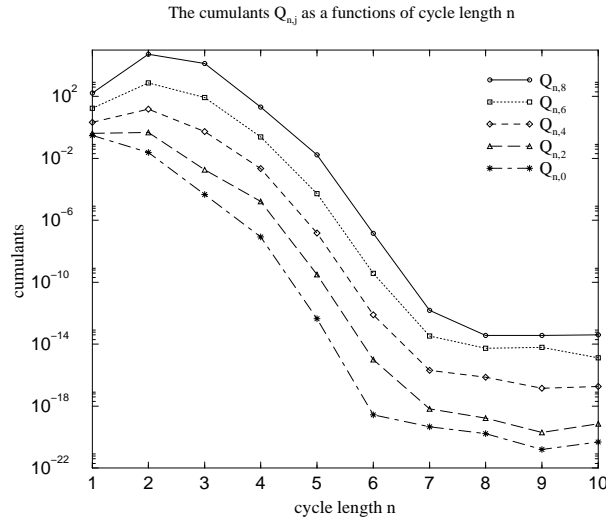


FIG. 1. The generalized cumulants $Q_{n,j}$ as a function of cycle length n for $j = 0, 2, 4, 6, 8$. $Q_{n,0}$ is the cumulant of the noiseless case. Superexponential convergence can be observed until cycle length $n = 6$, then numerical errors take over.

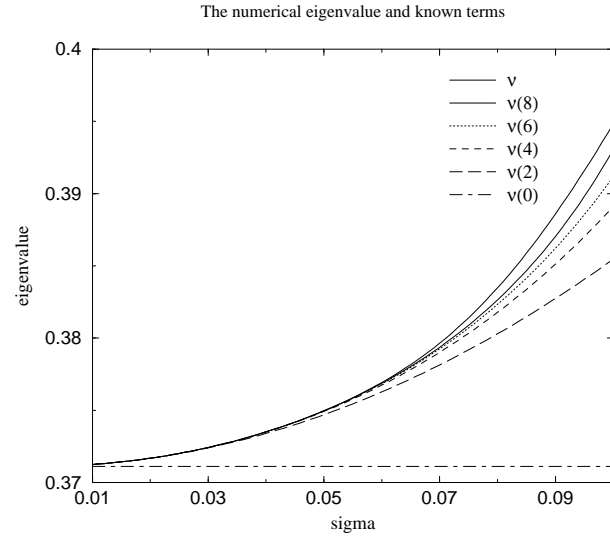


FIG. 2. The numerical eigenvalue and the known terms: $\nu(n) = \sum_{k=0}^{n/2} \nu_{2k} \sigma^{2k}$

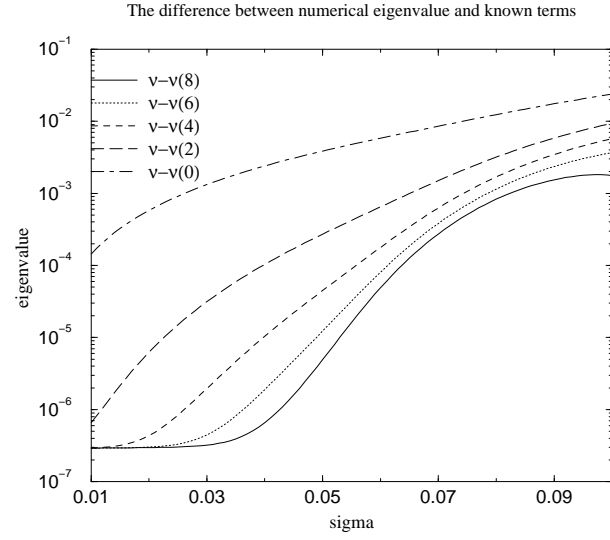


FIG. 3. The difference between the numerical eigenvalue and the known terms.

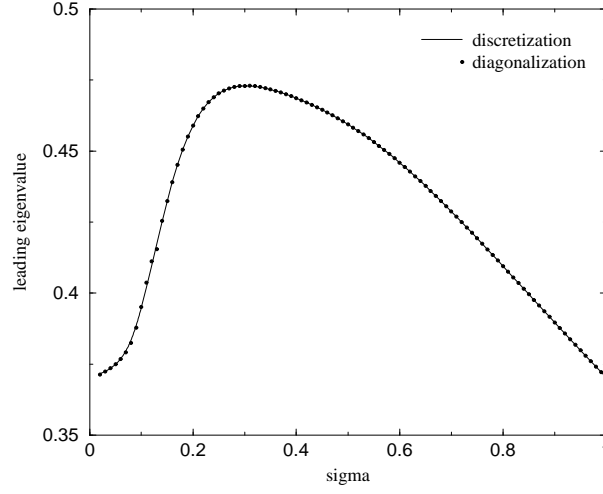


FIG. 4. The result of numerical integration and the eigenvalue fit together well in the large σ region.

VII. SUMMARY AND OUTLOOK

We have formulated a perturbation theory of stochastic trace formulas based on a polynomial basis matrix representations, expanded around infinitely many chaotic saddle points (unstable periodic orbits).

We note in passing that for 1- d repellers a diagonalization of an explicit truncated \mathcal{L}_{mn} matrix yields many more eigenvalues than the cycle expansions [16,10]. The reasons why one persists anyway in using the periodic orbit theory are partially aesthetic, and partially pragmatic. Explicit \mathcal{L}_{mn} demands explicit choice of a basis and is thus non-invariant, in contrast to cycle expansions which utilize only the invariant information about the flow. In addition, we do not know how to construct \mathcal{L}_{mn} for a realistic flow, such as the 3-disk problem, while the periodic orbit formulas are general and straightforward to apply.

VIII. ACKNOWLEDGEMENTS

G.V. thanks the Hungarian Ministry of Education, OMFB, OTKA T25866/F17166 and the Humboldt foundation the financial support. G. V. thanks Bruno Eckhardt the cordial hospitalty at the Department of Physics of the Philipps-Universität Marburg where some of the results have been derived. G.P. thanks the support of FKFP0159/1997.

APPENDIX A: ALGEBRA

To go up further in the powers of σ corrections in ν , we have to do the same as we have done for order four. First expand (42):

$$\begin{aligned} \det(1 - (z + z_0)\mathcal{L}) = & F - F_{10}z - F_{02}\sigma^2 - F_{20}z^2 - F_{12}z\sigma^2 - F_{04}\sigma^4 \\ & - F_{30}z^3 - F_{22}z^2\sigma^2 - F_{14}z\sigma^4 - F_{06}\sigma^6 \\ & - F_{40}z^4 - F_{32}z^3\sigma^2 - F_{24}z^2\sigma^4 - F_{16}z\sigma^6 - F_{08}\sigma^8 \end{aligned} \tag{A1}$$

The coefficients are:

$$\begin{aligned}
F_{30} &= \frac{1}{6} \sum_{m=3}^n \frac{m!}{(m-3)!} Q_{m,0} z_0^{m-3} & F_{22} &= \frac{1}{2} \sum_{m=2}^n \frac{m!}{(m-2)!} Q_{m,2} z_0^{m-2} \\
F_{14} &= \sum_{m=1}^n m Q_{m,4} z_0^{m-1} & F_{06} &= \sum_{m=1}^n Q_{m,6} z_0^m \\
F_{40} &= \frac{1}{24} \sum_{m=4}^n \frac{m!}{(m-4)!} Q_{m,0} z_0^{m-4} & F_{32} &= \frac{1}{6} \sum_{m=3}^n \frac{m!}{(m-3)!} Q_{m,2} z_0^{m-3} \\
F_{24} &= \frac{1}{2} \sum_{m=2}^n \frac{m!}{(m-2)!} Q_{m,4} z_0^{m-2} & F_{16} &= \sum_{m=1}^n m Q_{m,6} z_0^{m-1} \\
F_{08} &= \sum_{m=1}^n Q_{m,8} z_0^m
\end{aligned} \tag{A2}$$

The equation to be solved is:

$$\begin{aligned}
&F - F_{10}(z_2\sigma^2 + z_4\sigma^4 + z_6\sigma^6 + z_8\sigma^8) - F_{02}\sigma^2 - F_{20}(z_2\sigma^2 + z_4\sigma^4 + z_6\sigma^6 + z_8\sigma^8)^2 \\
&- F_{12}(z_2\sigma^2 + z_4\sigma^4 + z_6\sigma^6 + z_8\sigma^8)\sigma^2 - F_{04}\sigma^4 - F_{30}(z_2\sigma^2 + z_4\sigma^4 + z_6\sigma^6 + z_8\sigma^8)^3 \\
&- F_{22}(z_2\sigma^2 + z_4\sigma^4 + z_6\sigma^6 + z_8\sigma^8)^2\sigma^2 - F_{14}(z_2\sigma^2 + z_4\sigma^4 + z_6\sigma^6 + z_8\sigma^8)\sigma^4 - F_{06}\sigma^6 \\
&- F_{40}(z_2\sigma^2 + z_4\sigma^4 + z_6\sigma^6 + z_8\sigma^8)^4 - F_{32}(z_2\sigma^2 + z_4\sigma^4 + z_6\sigma^6 + z_8\sigma^8)^3\sigma^2 \\
&- F_{24}(z_2\sigma^2 + z_4\sigma^4 + z_6\sigma^6 + z_8\sigma^8)^2\sigma^4 - F_{16}(z_2\sigma^2 + z_4\sigma^4 + z_6\sigma^6 + z_8\sigma^8)\sigma^6 \\
&- F_{08}\sigma^8 = 0
\end{aligned} \tag{A3}$$

The solutions for z_6 and z_8 are:

$$z_6 = -\frac{1}{F_{10}}(2F_{20}z_2z_4 + F_{12}z_4 + F_{30}z_2^3 + F_{22}z_2^2 + F_{14}z_2 + F_{06}) \tag{A4}$$

$$\begin{aligned}
z_8 = -\frac{1}{F_{10}} &(F_{20}(2z_2z_6 + z_4^2) + F_{12}z_6 + 3F_{30}z_2^2z_4 + 2F_{22}z_2z_4 \\
&+ F_{14}z_4 + F_{40}z_2^4 + F_{32}z_2^3 + F_{24}z_2^2 + F_{16}z_2 + F_{08})
\end{aligned} \tag{A5}$$

The connection between the ν 's and the z 's:

$$\begin{aligned}
\nu_0 + \nu_2\sigma^2 + \nu_4\sigma^4 + \nu_6\sigma^6 + \nu_8\sigma^8 &= \frac{1}{z_0 + z_2\sigma^2 + z_4\sigma^4 + z_6\sigma^6 + z_8\sigma^8} \\
&= \frac{1}{z_0} \left[1 - \left(\frac{z_2}{z_0}\sigma^2 + \frac{z_4}{z_0}\sigma^4 + \frac{z_6}{z_0}\sigma^6 + \frac{z_8}{z_0}\sigma^8 \right) \right. \\
&\quad + \left(\frac{z_2}{z_0}\sigma^2 + \frac{z_4}{z_0}\sigma^4 + \frac{z_6}{z_0}\sigma^6 + \frac{z_8}{z_0}\sigma^8 \right)^2 - \left(\frac{z_2}{z_0}\sigma^2 + \frac{z_4}{z_0}\sigma^4 + \frac{z_6}{z_0}\sigma^6 + \frac{z_8}{z_0}\sigma^8 \right)^3 \\
&\quad \left. + \left(\frac{z_2}{z_0}\sigma^2 + \frac{z_4}{z_0}\sigma^4 + \frac{z_6}{z_0}\sigma^6 + \frac{z_8}{z_0}\sigma^8 \right)^4 \dots \right] \\
&= \frac{1}{z_0} - \frac{z_2}{z_0^2}\sigma^2 - \left(\frac{z_4}{z_0^2} - \frac{z_2^2}{z_0^3} \right) \sigma^4 - \left(\frac{z_6}{z_0^2} - 2\frac{z_2z_4}{z_0^3} + \frac{z_2^3}{z_0^4} \right) \sigma^6 \\
&\quad - \left(\frac{z_8}{z_0^2} - 2\frac{z_2z_6}{z_0^3} - \frac{z_4^2}{z_0^3} + 3\frac{z_2^2z_4}{z_0^4} - \frac{z_2^4}{z_0^5} \right) \sigma^8
\end{aligned} \tag{A6}$$

We can say that:

$$\begin{aligned}
\nu_6 &= -\frac{z_6}{z_0^2} + 2\frac{z_2z_4}{z_0^3} - \frac{z_2^3}{z_0^4} \\
&= (2F_{02}^3F_{20}^2 - 3F_{02}F_{10}F_{12}F_{20} + 2F_{02}F_{04}F_{10}^2F_{20} + F_{02}F_{10}^2F_{12}^2 \\
&\quad - F_{04}F_{10}^3F_{12} - F_{30}F_{02}^3F_{10} + F_{22}F_{02}^2F_{10}^2 - F_{02}F_{10}^3F_{14} + F_{06}F_{10}^4 \\
&\quad + 2(F_{02}^3F_{10}F_{20} - F_{02}^2F_{10}^2F_{12} + F_{02}F_{04}F_{10}^3)\nu_0 + F_{02}^3F_{10}^2\nu_0^2) \frac{\nu_0^2}{F_{10}^5} \\
\nu_8 &= -\frac{z_8}{z_0^2} + 2\frac{z_2z_6}{z_0^3} + \frac{z_4^2}{z_0^3} - 3\frac{z_2^2z_4}{z_0^4} + \frac{z_2^4}{z_0^5} \\
&= [F_{10}^4 (F_{08}F_{10}^2 - F_{06}F_{10}F_{12} + F_{04}F_{12}^2 - F_{04}F_{10}F_{14} + F_{04}^2F_{20}
\end{aligned} \tag{A7}$$

$$\begin{aligned}
& +F_{04}^2 F_{10} \nu_0) \\
& +F_{02}^4 (F_{04} F_{10}^2 - 5F_{20}^3 - 5F_{10} F_{20} F_{30} + 5F_{10} F_{20}^2 \nu_0 - 2F_{10}^2 F_{30} \nu_0 \\
& \quad + 3F_{10}^2 F_{20} \nu_0^2 + F_{10}^3 \nu_0) \\
& +F_{02}^2 F_{10}^2 (-3F_{10} F_{14} F_{20} + 6F_{04} F_{20}^2 - 3F_{10} F_{12} F_{22} + F_{10}^2 F_{24} \\
& \quad - 3F_{04} F_{10} F_{30} - 2F_{10}^2 F_{14} \nu_0 + 6F_{04} F_{10} F_{20} \nu_0 \\
& \quad + 3F_{04} F_{10}^2 \nu_0^2 + 6F_{12}^2 F_{20} + 3F_{12}^2 F_{10} \nu_0) \\
& -F_{02} F_{10}^3 (F_{13}^3 + F_{12} (-2F_{10} F_{14} + 6F_{04} F_{20} + 4F_{04} F_{10} \nu_0) \\
& \quad + F_{10} (F_{10} F_{16} - 2F_{06} F_{20} - 2F_{04} F_{22} - 2F_{06} F_{10} \nu_0)) \\
& -F_{02}^3 F_{10} (F_{10} (-4F_{20} F_{22} + F_{10} F_{32} - 2F_{10} F_{22} \nu_0) \\
& \quad + F_{12} (10F_{20}^2 - 4F_{10} F_{30} + 8F_{10} F_{20} \nu_0 + 3F_{10}^2 \nu_0^2)) \Big] \frac{\nu_0^2}{F_{10}^7}
\end{aligned} \tag{A8}$$

APPENDIX: REFERENCES

-
- [1] P. Cvitanović, C.P. Dettmann, R. Mainieri and G. Vattay, *Trace formulas for stochastic evolution operators: Weak noise perturbation theory*, *J. Stat. Phys.* (1998), to appear; **chao-dyn/9807034**.
 - [2] P. Cvitanović, C.P. Dettmann, R. Mainieri and G. Vattay, *Trace formulas for stochastic evolution operators: Smooth conjugation method*, submitted to *Nonlinearity* (1998); **chao-dyn/9811003**.
 - [3] P. Gaspard. *Chaos, Scattering and Statistical Mechanics*. Cambridge University Press, Cambridge, 1997.
 - [4] P. Cvitanović et al. *Classical and Quantum Chaos: A Cyclist Treatise*. Niels Bohr Institute, Copenhagen, 1998.
 - [5] P. Gaspard and D. Alonso Ramirez, *Phys. Rev.* **A 45**, 8383 (1992).
 - [6] P. Gaspard and D. Alonso. *Phys. Rev.* **A 47**, R3468 (1993).
 - [7] N.G. Van Kampen. *Stochastic Processes in Physics and Chemistry*. North Holland, Amsterdam, 1981.
 - [8] A. Lasota and M. MacKey. *Chaos, Fractals, and Noise; Stochastic Aspects of Dynamics*. Springer-Verlag, Berlin, 1994.
 - [9] D. Ruelle, *Ergod. The. and Dynam. Sys.*, 2:99, 1976.
 - [10] H.H. Rugh, *Nonlinearity*, 5:1237, 1992.
 - [11] P. Cvitanović, P.E. Rosenqvist, H.H. Rugh, and G. Vattay, *CHAOS* **3**, 619 (1993).
 - [12] D. Ruelle, *Statistical Mechanics, Thermodynamic Formalism* (Addison-Wesley, Reading MA, 1978).
 - [13] M.C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, New York 1990).
 - [14] R. Artuso, E. Aurell and P. Cvitanović, *Nonlinearity* **3**, 325 (1990).
 - [15] F. Christiansen, G. Paladin and H.H. Rugh, *Phys. Rev. Lett.* **65**, 2087 (1990).
 - [16] F. Christiansen, P. Cvitanović and H.H. Rugh, *J. Phys* **A 23**, L713 (1990).
 - [17] P. Cvitanović, ed., *Periodic Orbit Theory - theme issue*, *CHAOS* **2**, 1 (1992).
 - [18] P. Cvitanović, *Phys. Rev. Lett.* **61**, 2729 (1988).
 - [19] D. Ruelle, *Inventiones math.* **34**, 231 (1976).
 - [20] D. Fried, *Ann. Scient. Éc. Norm. Sup.* **19**, 491 (1986).
 - [21] R. Artuso, E. Aurell and P. Cvitanović, *Nonlinearity* **3**, 361 (1990).
 - [22] R.B. Dingle *Asymptotic Expansions: their Derivation and Interpretation* (Academic Press, London, 1973).
 - [23] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, (Dover, New York, 1964); Chapter 24., p. 823, formula I. B.
 - [24] L. Euler, *Opera Omnia* (Teubner, 1922).